Identifying Student Misconceptions with Formative Assessment Math Probes

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Cheryl Rose Tobey has 16 years of experience providing professional development to mathematics educators, primarily in the areas of diagnostic and formative assessments. She has coauthored 12 published Corwin books, including two volumes of the *Mathematics Formative Assessment: Practical Strategies for Linking Assessment, Instruction*, and *Learning* series, and seven books in the *Uncovering Student Thinking* series.

Cheryl is currently a mathematics consultant who provides supports and developed materials for mathematics educators across the country through her work for Education Development Center (EDC) and Maine Mathematics and Science Alliance (MMSA). While at EDC and MMSA, she led multiple National Science Foundation (NSF) and state-funded grants focused on increasing student achievement for struggling learners. Along with 10 years as a classroom educator, Cheryl has extensive experience coaching teachers and using math probes and other formative assessment techniques to better understand student learning.

Introduction

In today's classrooms, teachers are called upon to gather and use evidence of student thinking in a timely, formative way to implement and/or differentiate instruction that improves mathematics learning for all students. An important part of that process is "to identify and address potential learning gaps and misconceptions when it matters most to students, which is during instruction, before errors or faulty reasoning becomes consolidated and more difficult to remediate." (National Council of Teachers of Mathematics, 2014, p.53).

Powerful tools called Formative Assessment Math Probes are designed to support teachers in identifying student misconceptions. These math probes allow teachers to make sound instructional choices that are targeted at specific mathematics concepts and responsive to the needs of certain groups of students.

The Challenge of Misconceptions

Helping all students build understanding in mathematics is an important and challenging goal. Important steps in achieving this goal include gaining an awareness of student difficulties and the sources of those difficulties and designing instruction to diminish them (Yetkin, 2003).

Misunderstandings are a common difficulty and a normal part of learning mathematics. Many misunderstandings are overgeneralizations, that is, information extended or applied to another context in an inappropriate way. The following examples from the *Common Core Progression Documents* (http://ime.math.arizona.edu/progressions/) illustrate overgeneralizations:

Focus on Key Words: The language of comparisons can be difficult. For example, "Julie has three more apples than Lucy" means both that Julie has more apples and that the difference is three. Many students "hear" the part of the sentence about who has more, but they may not initially hear the part about how many more. Another language issue is that the comparison statement can be made in either of two related ways, using "more" or "less" *(K–5, Counting and Cardinality and Operations and Algebraic Thinking Progression)*.

Example response of a student with this misconception:

Sam has 7 apples, and Jack has 11 apples. How many more apples does Jack have?

"18 apples, because more means add."

Place Value: The decimal point is used to signify the location of the ones place, but its location may suggest there should be a "one-ths" place to its right to create symmetry around the decimal point *(K-5, Number and Operations in Base Ten Progression)*.

Example response of a student with this misconception:

What is the value of the 4 in the number 5.647?

"The four means 4 tenths in this number because it goes ones, tens, hundreds on both sides but with a ths instead of a just an s."

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Equality: Students who see equations written in only one way often misunderstand the meaning of the equals sign and think that the "answer" always needs to be to the right of the equals sign *(K–5, Number and Operations—Fractions Progression)*.

Example response of a student with this misconception:

$5 + 7 = x + 9$ "The x is 12 because $5 + 7$ is 12."

Intervals: Students sometimes have difficulty perceiving the unit on a number line diagram. When locating a fraction on a number line diagram, they might describe the unit as a portion of the entire number line. For example, when asked to show $\frac{3}{4}$ on a number line diagram marked from 0 to 4, a student might indicate the number 3 *(K–5, Operations and Algebraic Thinking Progression)*.

Example response of a student with this misconception:

Choose the letter that shows the location of $\frac{1}{2}$

□ A
□ B
□ C
□ D
□ E

"Point D is $\frac{1}{2}$."
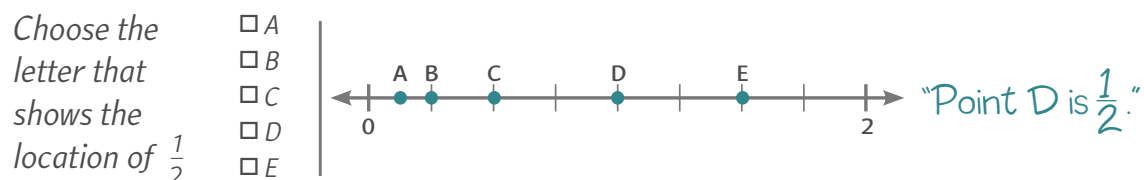
Expressions: Failure to see juxtaposition as indicating multiplication, e.g., evaluating 3x as 35 when x = 5, or rewriting 8–2a as 6a *(6–8, Expressions and Equations Progression)*.

Example response of a student with this misconception:

Find the value of 4m + 8 when m = 3   "$43 + 8 = 51$"

Function Notation: Students sometimes interpret the parentheses in function notation as indicating multiplication. Because they might have seen numerical expressions like 3(4), meaning 3 times 4, students can interpret $f(x)$ as f times x. This can lead to false generalizations of the distributive property, such as replacing $f(x + 3)$ with $f(x) + f(3)$ *(High School, Functions Progression)*.

Example response of a student with this misconception:

Evaluate $f(x) = 3x + 9$ when x = 7

"$f(7) = 21 + 9$ simplify this to get $f(7) = 30$ Divide by 7 So, $f = \frac{30}{7}$."

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It is important to note that a misconception begins with flawed conceptual understanding and can lead to errors in mathematical application. Mathematical errors can and do occur because of misconceptions, but this is not true of all mathematical errors. For example, a copying mistake or simple calculation error within a large, multiple-step process are errors that are not caused by misconceptions.

The topic of rational numbers further illustrates the challenges of mathematical misconceptions. Many students' difficulties and errors with rational number concepts derive from common conceptual flaws. These flaws can stem from the inherent properties of rational numbers and the transition from learning about and working with the whole number system. While whole number relationships are based on additive properties, rational numbers have relationships based on multiplicative relations. Moreover, rational numbers can be expressed in many different forms (e.g., common fractions and decimal fractions), they use a new system of symbols, and they can be designated by an infinite number of equivalent representations.

When rational numbers are introduced, two numbers that students previously understood as functioning independently now compose a new number (e.g. a common fraction) that has a distinct value. To add further complication, a single rational number can be representative of several distinct conceptual meanings. These distinct conceptual meanings are referred to as "sub-constructs" of a rational number. These sub-constructs include: part-whole relation (4 of 5 equal shares); quotient interpretation (implied division in which 2 submarine sandwiches are divided for 3 boys); measure (fixed quantity on a number line); ratio (5 girls to 6 boys); and multiplicative operator (scaling: reduce or enlarge) (NRC, 2005).

Many students do not understand the meaning of the fraction symbol, and instead focus on either the numerators or denominators when ordering or comparing common fractions. When comparing a fraction such as $\frac{5}{7}$ to $\frac{4}{9}$, they will compare 7 and 9. Students with this misconception may conclude that since 9 is greater than 7, then $\frac{4}{9}$ is greater than $\frac{5}{7}$. Students also hold common misconceptions about the addition of rational numbers. They may add the two fractions $\frac{1}{5}$ and $\frac{1}{4}$ and think the solution is $\frac{4}{9}$ (because they add the numerators together and they add the denominators together).

Often students have difficulty with the fact that the two numbers (numerator and denominator) composing a common fraction are related through multiplication and division, not addition. This may create problems when comparing fractions, because students look at the difference between numerator and denominator to come to an inaccurate conclusion. When looking at $\frac{7}{8}$ and $\frac{5}{6}$, they may see that in each case the difference between numerator and denominator is one, wrongly concluding that these two fractions are equal. In each of these examples, the difficulties described stem from a flaw in the student's conceptual understanding of rational numbers.

It is impossible to teach in a way that avoids creating any misconceptions, so we should accept that students will make some incorrect generalizations that will remain hidden unless the teacher makes specific efforts to uncover them (Askew & Wiliam, 1995). A teacher's role is to minimize the chances of students harboring misconceptions by acknowledging potential difficulties, using assessments to elicit misconceptions, and implementing instruction to help students build conceptual understanding of the mathematics.

The Power of Formative Assessment Math Probes

A Formative Assessment Math Probe is a short, highly-focused, quick-to-administer diagnostic assessment designed to pinpoint specific misconceptions students may have about a mathematical concept. Math probes have been developed, field-tested, and implemented by teachers for more than 10 years (Keeley & Rose-Tobey, 2006, 2011, 2017). Along with collections created for specific grants, published sets of math probes (like those in the series *Uncovering Student Thinking*) offer specific, grade-level assessments that promote deep learning (Rose, Minton, & Arline, 2007; Rose & Arline, 2009; Rose & Minton 2010; Rose-Tobey & Arline, 2014; Rose-Tobey & Fagan, 2013, 2014).

Math probes from all resources have important common characteristics. Each math probe typically includes three to six items, and each item requires a two-part response from the student: a selected response and a written explanation using words and/or pictures. Together, this combination helps to reveal underlying patterns in incorrect answers and will show whether correct selected responses are supported by strong or by faulty reasoning. Multiple items targeting a specific topic provide important insights into why a student may be having difficulty.

Following are two examples of math probes:

Estimating with Percent

Without calculating, determine the best choice for an estimate.

Circle your choice:	Explain your choice(s):
1) What is 5.23% of 61.15 a. 1200 b. 300 c. 120 d. 30 e. 12 f. 3	
2) What % of 3.18 is 17.9 a. 1600 b. 600 c. 160 d. 60 e. 16 f. 6	
3) 0.41% of what is 21.84 a. 5000 b. 2000 c. 500 d. 200 e. 50 f. 2	

Name _____

Multiplication Equations

	Explain how you got your answer.
A) $6 \times 4 = \square \times 3 \times 2$ Circle the number that belongs in the box: **2 4 6 24**	
B) $9 \times 6 = 6 \times 3 \times \square$ Circle the number that belongs in the box: **3 6 18 54**	
C) $6 \times 8 = (\square \times 8) + (2 \times 8)$ Circle the number that belongs in the box: **3 4 16 24 48**	
D) $8 \times 9 = (\square \times 9) + (\square \times 9)$ Circle the number that belongs in the boxes: **4 8 17 18 72**	

Multiple selected response items elicit understandings related to the meaning of the equals sign.

Multiplication Equations

Name _____

A)

$$6 \times 4 = \square \times 3 \times 2$$

Circle the number that belongs in the box:

2   4   6   24

Explain how you got your answer.

B)

$$9 \times 6 = 6 \times 3 \times \square$$

Circle the number that belongs in the box:

3   6   18   54

Explain how you got your answer.

C)

$$6 \times 8 = (\square \times 8) + (2 \times 8)$$

Circle the number that belongs in the box:

3   4   16   24   48

Explain how you got your answer.

D)

$$8 \times 9 = (\square \times 9) + (\square \times 9)$$

Circle the number that belongs in the boxes:

4   8   17   18   72

Explain how you got your answer.

Multiple justification prompts provide additional evidence of student thinking.

"Before using math probes, I was aware that some concepts were difficult for students year after year, but I wasn't always digging deep enough into the reasons why these concepts were difficult. Training on implementing math probes and analyzing student work has helped me become misconception-conscious."

– Grade 5 Teacher

Math Probes and Formative Assessment

A 1998 review of more than 250 articles related to formative assessment highlights the importance of formative assessment in learning and teaching (Black and Williams, 1998). Since then, researchers and practitioners have investigated the topic more deeply and clarified the attributes of formative assessment, resulting in an expanded definition. Formative assessments are currently defined as the systematic and iterative process of gathering and analyzing evidence, providing feedback to students, altering instruction, and gathering more evidence to determine if the resulting instructional actions moved students' learning forward.

Brief descriptions of critical aspects of the formative assessment process are summarized in Table 1 (Creighton, Tobey, Karnowski and Fagan, 2015).

Table 1: Critical Aspects of Formative Assessments

Critical Aspects of the Formative Assessment Process	
Learning Targets	Goals that articulate the learning that a teacher intends to happen and indicate to both the teacher and student whether that learning is taking place
Evidence	A process of gathering information about student thinking and student skill and interpreting it against success criteria to determine next instructional steps
Responsive Action	A process of determining the appropriate next instructional steps to help students move their learning forward. These actions might include: ■ gathering more evidence of students' thinking to gain clarity about appropriate next steps; ■ providing "formative feedback" that is aligned to the learning target; ■ providing further instruction if feedback will be insufficient to help a student move forward; and ■ determining that the student has met the learning target and moving on to a new target
Student Ownership and Involvement	A set of strategies to provide students with the skills they need to become self-regulating learners and to use each other as peer resources

Math probes are one strategy for eliciting evidence of student thinking within this formative assessment framework. Since these components of formative assessment are to be implemented while learning is underway, the use of math probes can be considered a short-cycle intervention. Intervention in the formative assessment framework can be described as full-class or small-group instruction in a classroom setting or within an intervention setting: It uses the results of evidence to drive instruction and address specific concepts within a learning target.

"Using math probes has helped me to analyze student work in a different way. By looking for patterns in thinking of both correct and incorrect selected responses, I am often surprised in both directions—what they know and what they don't know."

– Grade 2 Teacher

Math Probes and Responsive Action

While diagnostic assessment involves learning about student thinking, formative assessment involves using what you've learned about students' understandings to design and implement instructional experiences. Math probes are considered diagnostic until a teacher acts on the data to move student learning forward. When teachers are clear about the conceptual understanding they are working to build, and they have analyzed the math probe results, they are well situated for the next step in the process: moving from diagnostic to formative assessment by providing targeted learning activities.

"The phrase 'to teach without causing misconceptions is impossible' really resonates with me. Using math probes helps me learn how students have interpreted my instruction and the mathematics activities I have asked them to engage in. I can then more easily pinpoint additional experiences that will support their learning."

– Algebra II Teacher

Consider the following classroom vignette:

To get a sense of what my students understood from our initial lessons on fraction addition, I gave a math probe on estimating fraction sums. In reviewing my students' responses to the math probe, I noticed the following:

- *A few students approached the problem by estimating percent equivalents and adding those to determine whether the sum was more or less than the benchmark. They used this strategy effectively, and many used this strategy across each of the problems on the math probe.*

- *A handful of students misapplied whole-number thinking, such as adding the numerators and adding the denominators to find the sum of the fractions. These students did not use estimation at all.*

- *Another group of students did not estimate but rather correctly applied the addition algorithm to find the sum of the fractions. While these students got correct answers, their lack of estimation raised questions for me about their ability to reason about the size of the fractions and the operation of addition.*

- *A few students used varied estimation strategies to choose the correct selected response, and they supported their selection with solid reasoning.*

By looking for patterns in understandings and misunderstandings, I was better able to pinpoint the types of problems students were having, allowing me to explore and discuss them in the upcoming lessons (Tobey and Fagan, 2014, p.186-187).

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In this example, a high school teacher describes her process of sorting through student responses to a math probe.

About three-quarters of the way through the unit, I gave the probe on matrices to determine the extent to which my students were beginning to generalize from examples and nonexamples the various properties of matrix operations. I gave the math probe as an exit ticket and sorted the responses to get a sense of the class. I started with #1, and first quickly sorted the papers into two piles, True and False. From there, I quickly sorted those with False into three categories: those who had solid justifications, those whose justifications didn't provide evidence of solid understanding, and those whose justifications showed evidence of correct response for an incorrect reason. I also sorted those with True into two categories: those whose justifications showed some correct reasoning and those who showed the overgeneralization from number properties that all multiplication is commutative (see Figure 1).
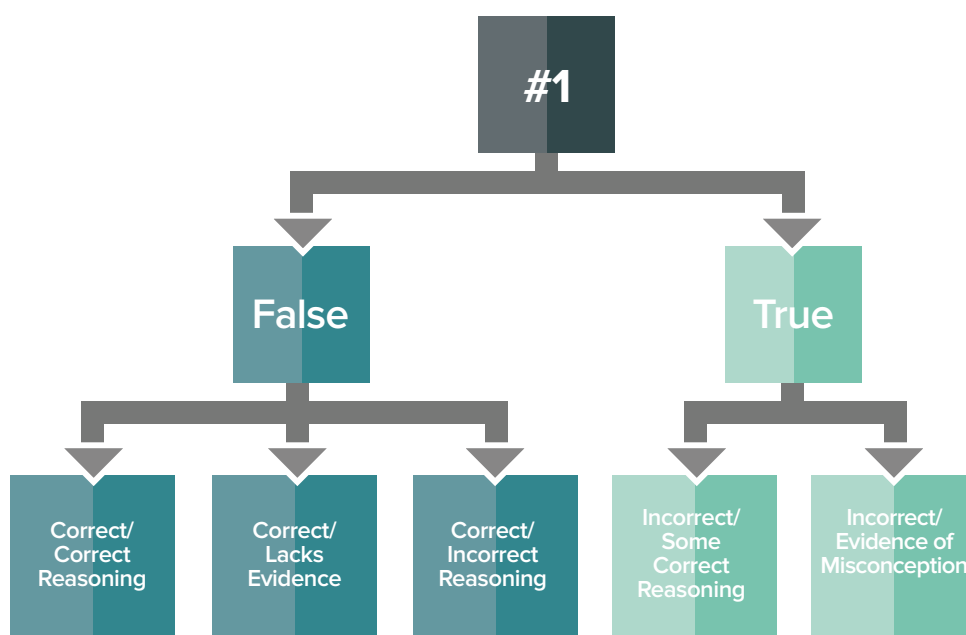
```
                          #1
            ┌──────────────┴──────────────┐
          False                          True
    ┌───────┼───────┐              ┌──────┴──────┐
 Correct/ Correct/ Correct/   Incorrect/   Incorrect/
 Correct  Lacks    Incorrect  Some         Evidence of
 Reasoning Evidence Reasoning  Correct      Misconception
                               Reasoning
```

Figure 1

After making some notes about the data, I continued working through the other problems using a similar process. After determining that more than half of the class either had incorrect reasoning or insufficient evidence on each of the problems, I decided on the following learning goal for the next lesson:

- ■ *Learning Target: How matrix properties are similar to and different from rational numbers properties*

- ■ *Success Criterion 1: Use matrix properties to justify statements that are true*

- ■ *Success Criterion 2: Use a counterexample to justify a false statement*

To help students meet these criteria, I created and used a series of four short activities using examples and nonexamples, technology, and readings (Tobey and Arline, 2014, p. 199).

Both vignettes showcase how teachers elicit and interpret student responses to a math probe to make instructional decisions.

When and How to Use Math Probes

Teachers need to be clear with students about the purpose of the math probe and how they will use the information gathered. As noted previously, a math probe looks similar to a traditional quiz or summative assessment. However, because they are formative tools, math probes are not meant to be graded. For students who are not accustomed to explaining their thinking, a math probe can be challenging. We must support students in building this skill over time. Consider comparing samples of explanations (correct and incorrect) that show thinking along with explanations (correct and incorrect) that do not. Students who have difficulty writing may need to dictate their thinking, or you can conduct a follow-up interview with the student to get a fuller picture of their understandings/misunderstandings.

"At first I thought students wouldn't give me their best effort on a task that wasn't going to 'count.' By talking with them about the purpose of a math probe and how the results would be used, I was pleasantly surprised to hear that students appreciate the idea of the math probe cycle: Take math probe; Discuss data and engage in additional learning tasks; Revisit math probe; Discuss growth."
– Grade 7 Teacher

Answers to Frequently Asked Questions

When do I administer a math probe?

A math probe can be administered at different times during a sequence of instruction to achieve different purposes:

- before a unit of study to assess prerequisite knowledge and understandings and to target intervention prior to the unit of study; or

- during instruction as a formative assessment of progress toward learning goals and to plan next steps in instruction; or

- after a unit of instruction to assess progress/proficiency in learning goals

To whom do I give the math probe?

A math probe can be administered to a whole class or to a subset of students in the class. Some options include delivering the math probe:

- in written form to all the students in your class to analyze class patterns; or

- to a subset of students to assess progress resulting from a targeted intervention or to learn more about why those students may be having difficulty; or

- in interview form to a subset of students to learn more about how students at various levels are thinking about a topic

What do I do after collecting the math probe?

After your students have completed the math probe, you will:

- sort your student work to determine categories of understandings and misunderstandings;

- use this information from the sort to determine instructional next steps; and

- determine how targeted instruction affected your students' thinking by:

    (1) administering an alternate version of the original math probe; or

    (2) returning students' original work on the math probe to them and inviting them to revise their thinking/reflect on how their thinking has changed; or

    (3) creating a short exit ticket with one or two math probe items and a reflection prompt; or

    (4) differentiating the post assessment by giving a more challenging math probe for students who showed strong understanding on the initial administration. (Fagan, Tobey, & Brodesky, 2017)

Math Probes and the ACT Cycle

The teacher support materials that accompany Formative Assessment Math Probes are designed around an ACT cycle. The ACT cycle was originally developed during the creation of a set of math probes and teacher resources for a Mathematics and Science Partnership Project in Oklahoma. It consists of three teacher actions around using math probes:

1. **Analyze the Math Probe**

   Prior to administering the math probe, the teacher completes the math probe items and anticipates student difficulties.

2. **Collect and Assess Student Work**

   After administering the math probe, the teacher reviews students' selected responses and explanations to look for patterns of understandings and misunderstandings.

3. **Take Action**

   After assessing student thinking, the teacher determines next instructional steps.

Summary

Math probes are tools that enable teachers in all grades to gather important insights into erroneous mathematics thinking in a practical way. Learning about common student difficulties and misconceptions and their root causes helps teachers to chunk the learning into systematic and manageable pieces. Math probe evidence won't tell a teacher everything about a student's understanding of a concept, such as "fractions;" however, a math probe can yield a right-sized amount of information about conceptual understanding, allowing a teacher to act strategically to move learning forward (Fagan, Tobey & Brodesky, 2017).

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