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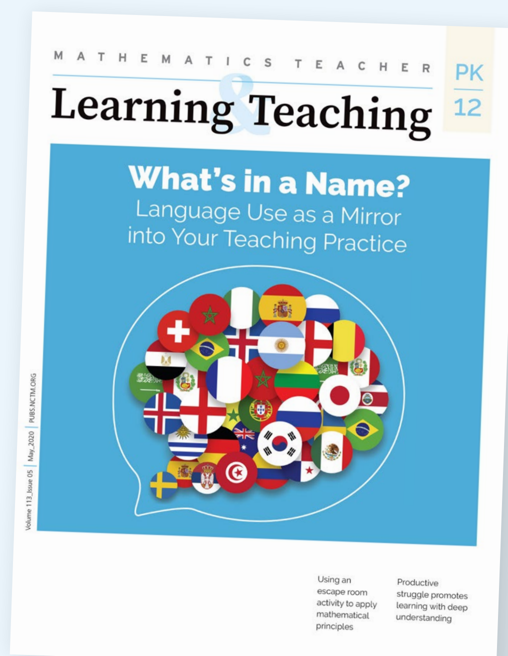

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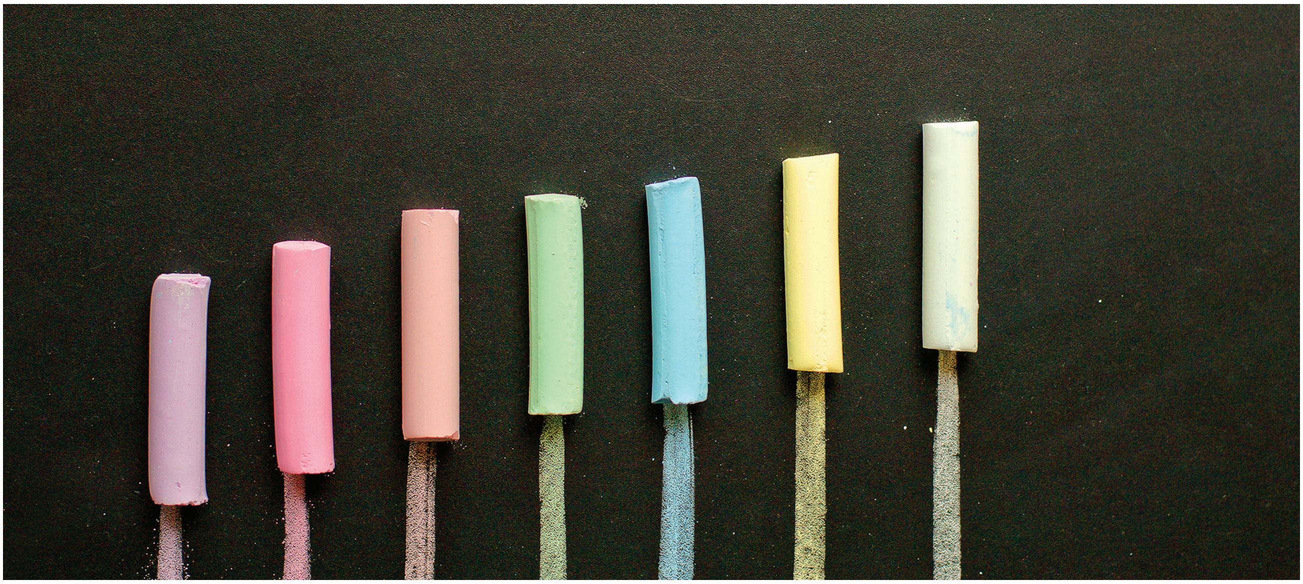


Challenging but Achievable Math for Young Children

Learning trajectories help teachers challenge children at just the right level for their best learning.

Douglas H. Clements, Shannon S. Guss, and Julie Sarama

Ms. Farrauto presented a tangram puzzle for two of her preschoolers. Soma figured out how to slide and turn most shapes into position to make the rectangle. But he was challenged by one shape, a parallelogram (see figure 1a).

Soma: I know it's the right shape, but it doesn't go in.
Tiara: You gotta flip it over.
Soma: Like this? Oh, look, got it!" [*See figure 1b.*]

Previously, Ms. Farrauto had noticed Soma's ease with composing shape puzzles through slides and turns. So, she used her knowledge of the path—the developmental progression for composing shapes—to challenge him with a new puzzle that required him to flip shapes.

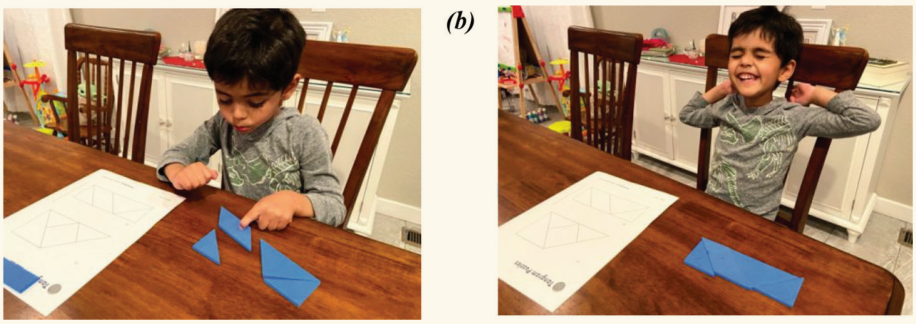
Ms. Farrauto noticed and understood her student's mathematical thinking and used formative assessment to guide his learning to the *next* level of thinking. She knew that teaching based on children's thinking increases *all* children's engagement and learning *and* the joy of teaching (Jacobs, Lamb, and Philipp 2010; Lee and Santagata 2020; Luebeck and Lindsay 2021). Fortunately, *learning trajectories* (LTs) are an ideal tool for such formative assessment (Clements and Sarama 2021; Frye et al. 2013; NCTM 2018; Sarama and Clements 2009; Wilson, Mojica, and Confrey 2013). Unfortunately, many PK to grade 3 children and teachers do not have access to research-validated early mathematics resources, especially those based on LTs (Clements and Sarama 2021). The purpose of this article is to describe and illustrate LTs and introduce a new, comprehensive set of resources [link online] that support teaching as meaningfully and joyfully as for Ms. Farrauto.

LEARNING AND TEACHING WITH LTS

Teaching such as Ms. Farrauto's requires understanding (1) standards and specific goals; (2) the ways children think and learn about mathematics; and (3) effective ways to teach, supporting children's ways of learning (Clements and Sarama 2021). That is why an LT for a topic has three parts: (1) a mathematical goal, (2) a developmental progression of levels of thinking, and (3) teaching practices (environments, activities, hints, and scaffolds) that challenge children at just the right level (Clements and Sarama 2021; Sarama and Clements 2009). Fortunately, a powerful new resource, the Learning and Teaching with Learning Trajectories tool (link online) is designed to help early childhood educators, professional development providers, and families learn how children from birth to third grade think and learn about mathematics as well as how to support that learning (we use the initialism LTLT or [LT]^2—we admit, one of those mathematics "jokes" with limited humor value). Further, [LT]^2 provides materials

Fig. 1

(a) *(b)*



Examining (a) a trapezoid shape, preschooler Soma puzzles through (b) to a solution.

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for teaching and learning, all at [LT]^2 (link online; you may wish to pause reading and explore it briefly). When we teachers understand mathematical content and how children think and learn to develop their understanding of mathematics, we can notice and create teachable moments throughout the day. Further, hundreds of intentional activities are provided in [LT]^2, fine-tuned to each level of thinking you wish to help your children develop.

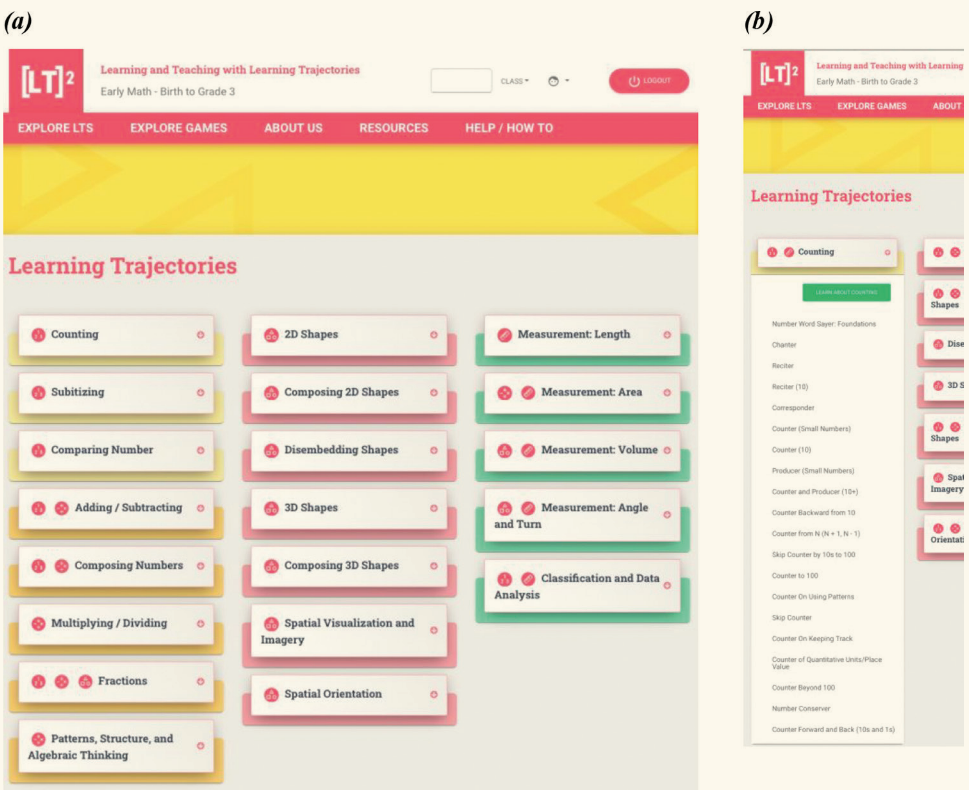
A Journey through One LT

Although LTs are particularly useful for less well-known topics, such as Soma's work on shape composing and spatial visualization, Ms. Farrauto found the LT approach surprisingly important when considering *familiar* topics, such as counting.

Goal. Ms. Farrauto decided to study the LT for counting because she was a bit taken aback that some children did not do well on the school's assessment. From the 20 topics on [LT]^2 (see figure 2a), she selected Counting (see figure 2b). Next, she selected the green "Learn About" button and discovered that an LT's goal is more than a simple behavioral objective. Instead, counting includes mathematical concepts, skills, processes, and connections across *verbal* counting, *object* counting with understanding, and even *strategic* counting for solving various problems (Clements and Sarama 2021). Children's patterns of thinking across levels, especially higher levels, became more apparent as she worked her way through the developmental progression.

As Ms. Farrauto worked through the descriptions and videos at each level of the developmental progression, she learned that even verbal counting should be mathematically meaningful—not rote. For example, children showing behaviors consistent with the [LT]^2 level called *Chanter* may say the counting words in order but run the words together, such as "onetwothreefour" (like "LMNOP"). Children then develop the ability to say each number word separately: [LT]^2's Reciter level. Although

Fig. 2



From 20 topics on learning trajectories for each important topic in early mathematics on (a) the [LT]^2 website, Ms. Farrauto opened (b) the one for Counting.

they do not link counting words to a given quantity yet, even at these early levels, counting is not rote; that is, verbal counting is not meaningless memorization of words. For example, children do not mix number words with other words or letters. Most Reciters also implicitly understand that the further they count, the "bigger" the number is (appropriately, they have no such notion when reciting the alphabet). These initial intuitions are meaningful and important.

Next, Ms. Farrauto viewed [LT]² videos and read descriptions of levels for object counting. These showed that even if children could recite to 10, they might point at objects generally but not accurately keep one-to-one correspondence. First, children must maintain focus and concentration (see the Corresponder level in [LT]²) because object counting demands *two* one-to-one correspondences. Children must say each counting word in correspondence with their pointing acts *and* correspond these pointing acts with the things they are counting. This is more challenging than the simple task of object correspondence, such as giving one snack to each child at a table. Second, when objects are in random positions and cannot be moved, children need spatial abilities to remember which have been and have not yet been counted (see the Counter and Producer [10+] level on [LT]²).

So, meaningful verbal counting and challenging one-to-one correspondence are two necessary competencies in object counting. But that is not all. Ms. Farrauto learned about a third, oft-neglected competence. (Can you think what it is?)

When Ms. Farrauto learned what it was, her first thought was, "I never knew we had to teach that." The third competence is understanding that the last number word one says when counting tells *how many* objects are in the set. This cardinality principle is obvious to adults, so it is easy to miss this major developmental advance for children (it defines the Counter [Small Numbers] level). Corresponders who do not yet understand the cardinality principle might, as in one video you can see on [LT]², count one, two, three, four (correctly) and when asked how many, they enumerate again one, two, three, four, as if the very act of counting was their answer. If the four objects are covered when asked how many, other Corresponders say five or ten or one hundred. Ms. Farrauto concluded that the act of counting at the early levels is first about keeping things in order—keeping the number words in order, keeping the objects in order and in correspondence with the number words, and so forth.

Then, when you are finished counting, the third competence is making a count-to-cardinal transition (Sarama and Clements 2009). That is, you switch from one-by-one counting to thinking of *how many are in the group.* Indeed, if a group has *four,* it does not matter if you start at one end with one . . . , counting left to right, or start at the other end, counting right to left; the answer is always four. The particular order disappears, transformed into *how many*, or *the cardinal number in the group.* A cognitive leap. A new level of thinking. Counting has (finally) produced a mental quantity (see figure 3, which links online to guided instructions for accessing multiple resources and videos of this and other levels in [LT]²). Ms. Farrauto told us, "Just knowing this has changed the way I understand and interact with children when they're counting." [LT]² can enrich your observations and interactions too.

For the children in her class who had already demonstrated these competencies, Ms. Farrauto noted that they could soon reach later levels connected to addition and subtraction. For example, one counting competence involves counting that starts at numbers other than one (Counter from N ($N + 1$, $N - 1$) on [LT]²) and even counting on from a small number ("What comes two numbers after four?" "Foooour . . . five, six. Six."; see Counter On Using Patterns). Also, children at all levels can use strategic counting to solve problems—from getting enough pencils for their table, to making sure everyone received the same number of grapes, to figuring out how many *more* a child needs so they have as many as their friends.

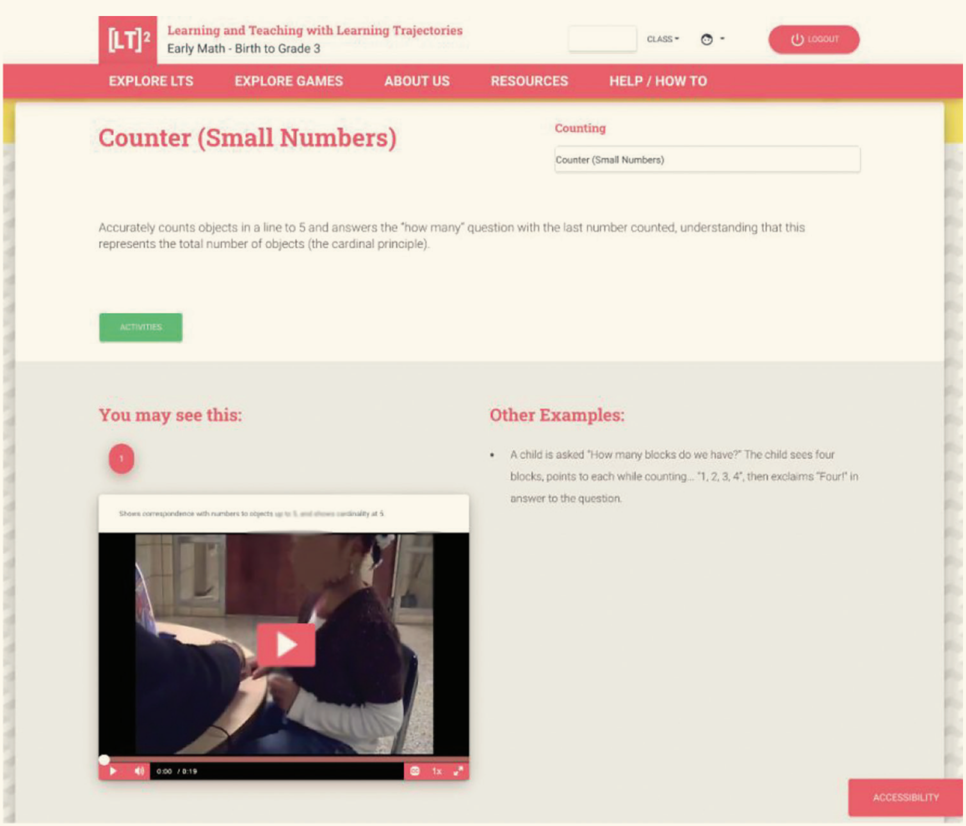
Instructional strategies. The third part of a learning trajectory is teaching strategies, including environments, interactions, and activities. The [LT]² site organizes teaching strategies tailored to each successive level of children's understanding. For example, to help her students who need to move from verbal counting to the Corresponder level, Ms. Farrauto engaged them in an [LT]² online activity, Get Ready to Go, which shows a bird's-eye view of an empty bus with 10 seats (a ten-frame) and some children waiting in the parking lot beside the bus. Tapping on one child icon at a time gives each a hat, marking them as counted. Simultaneously, each counting word is spoken by the computer. This scaffolds students' work, allowing them to focus *just* on the one-to-one correspondence. Once all virtual children are counted, they are loaded onto the bus, and the bus drives away. The game also provides feedback (Clements and Sarama 2021; Luebeck and Lindsay 2021): If users try to select a child twice, they are told they have already counted it. If they missed tapping on any children, the bus does not go, and they get a reminder. In the classrooms' mathematics centers, students work on a similar activity—putting toy children into a pictured bus—working in pairs and checking each other's work. Compare this to a counting activity in which teachers lay out some objects and then model saying the counting words and keeping one-to-one correspondence. In contrast, the Get Ready to Go game puts the student in a more active role, which is a more effective teaching strategy.

Once students could correspond, at least with objects in a row, Ms. Farrauto knew her teaching strategies must change. Her previous classroom practice just repeated "laying out and counting objects." To teach the even more abstract notion of cardinality, [LT]² suggested several playful activities (see figure 4). Why are they more effective? Consider one Ms. Farrauto selected, How Many in a Hand?

This [LT]² activity focuses on cardinality (that counting tells "how many"). Ms. Farrauto started by telling children she saw some wooden inch cubes and thought, "I wonder how many I can hold in one hand?" She explained she had as many as one of her hands could hold, now hidden behind her back. She asked children to help her count to find out how many cubes she had hidden in her hand. She removed one block, keeping the rest hidden, placing it in front of the children, so they see and focus on this one and chorally count "one." She repeated this until she had counted out all cubes. Displaying her now-empty hands, she gestured around all the cubes and asked the children how many there were in all. Gesturing again, she confirmed, "Four. We counted, and there are four."

Why is this effective? First, unlike the usual counting activity, starting by hiding cubes immediately makes children curious about cardinality as they wonder, "How many are back there?" Second, notice that children hear each counting word as it is spoken in enumeration while

Fig. 3



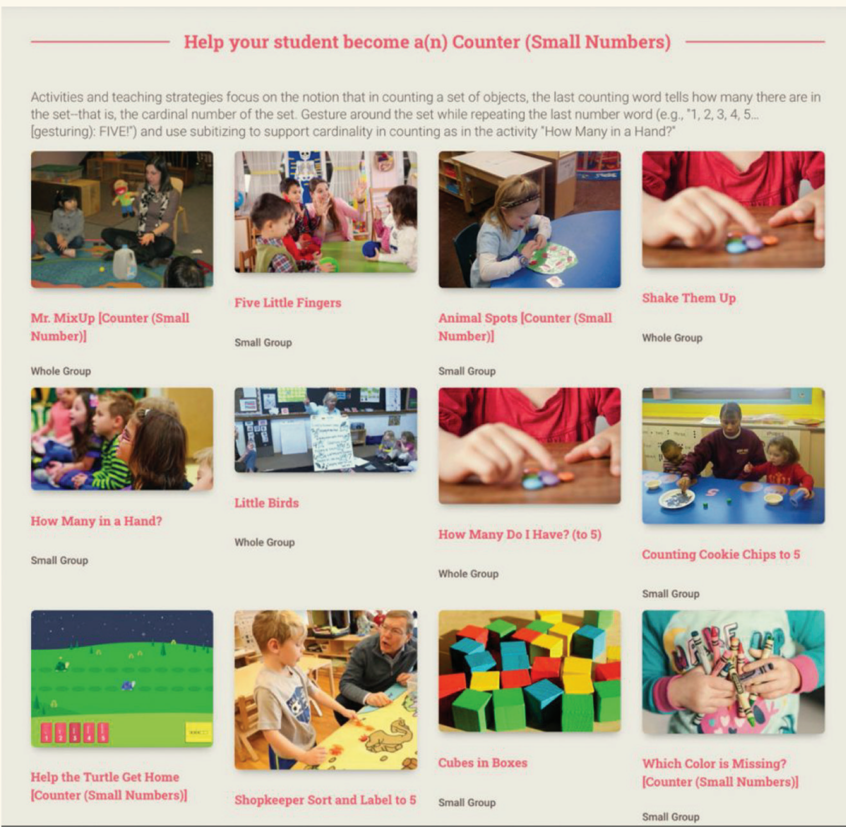
The figure shows one level in the Counting learning trajectory. Online (link online), you can connect to guided instructions for using the [LT]² site and videos.

observing the corresponding collection containing that number of objects—and most children can recognize very small numbers (especially if you have already helped them to *subitize*—that is, quickly recognize without counting—featured on [LT]²). So, a child who does not yet know the cardinality principle will notice: When we say "one," I see one. When we say "two," I know that's two, and so forth. So by "four," that idea has been clearly illustrated (that is why we bring the cubes out one at a time, instead of pointing to them in a line that is always visible). Third, gesturing around them and repeating that "there are four" reinforces that critical idea. Thus, this activity incorporates (1) the goal of teaching cardinality with (2) an understanding that children have recently constructed correspondence concepts (the developmental progression) and (3) instructional strategies of engaging students' curiosity, visually connecting concepts of correspondence and enumeration, with intentional use of gesture to emphasize cardinality.

The activity does not end there. Ms. Farrauto challenged children to see how many fit in *their* hands, asking, "Do you think you can hold as many in one hand?" After discussion, she challenged them: "I think your hand is smaller; I don't think you can hold as many as me." Now they are even more actively engaged.

Are there other characteristics that make this activity better than simply laying out objects and pointing while counting? Yes, it *starts with a cardinal question*, "How many in my hand?" that counting *answers*—contextualizing the activity with a problem that children can solve with mathematics. Also, asking children to do it themselves is more motivating and meaningful (and, you can probably guess, this will lead to counting higher than four, as children struggle to beat you). This is one example of how [LT]² can help teachers understand a developmental progression and provide meaningful and playful activities that have been tested in classrooms and tailored to specific levels of the developmental progression.

Fig. 4



The [LT]² site shows instruction suggestions for the level of Counter [Small Numbers].

The Impact of Learning Trajectories

Children who experience teaching grounded in learning trajectories learn more and deeper mathematics than those using other curricula. The developmental progressions in [LT]² are based on an integrated synthesis of thousands of studies (Clements and Sarama 2021; Sarama and Clements 2009). More importantly, an extensive series of longitudinal studies of hundreds of teachers and thousands of students confirm that children learn more using the learning trajectories approach than other approaches (Baroody, Clements, and Sarama 2022; Clements et al. 2019). They amaze their teachers with what they learn to do in mathematics (Clements and Sarama 2008; Clements et al. 2011; Clements et al. 2013). The strong, reliable benefits of the learning trajectories approach have been reported in other countries as well, such as Ecuador (Bojorquea et al. 2018).

Finally, research has documented race and class inequities in opportunities to learn stemming from deficit-thinking and income disparities (McKown and Weinstein 2008). Resources in [LT]² include school and out-of-school guides to equitable education. Perhaps most important, culturally relevant practice begins with high expectations and affirming relationships and is supported by teachers' continual reflection on children's cultural and individual identities. We educators must observe and interpret children's thinking, checking our own biases about children's race, ethnicity, language, and ways of expressing knowledge. Beginning with an expectation that all children can learn, we can develop the ability to notice and interpret children's diverse ways of expressing mathematical reasoning using developmental progression. The scale-up studies found that Black children benefited the most from the learning trajectories approach (Clements and Sarama 2008; Clements et al. 2011; Clements et al. 2013). Further, a follow-up study found that teachers using learning trajectories from [LT]² had higher expectations of children and engaged them in higher-level mathematical thinking, with the most significant benefits for Black children (Clements et al. 2011; Schenke et al. 2017).

CONCLUSION

Learning trajectories are important because young children's ideas and interpretations of situations are different from those of adults. Therefore, we teachers must interpret what the child is doing and thinking, attempting to see the situation through the child's eyes. Knowledge of developmental paths can enhance your understanding of children's thinking, helping you assess children's levels of understanding and offer instructional activities at those levels. Similarly, you can better see instructional strategies from the child's perspective, offering meaningful and joyful opportunities to engage in learning.

Understanding children's thinking and development ensures we meet them where they are and move them forward. The LT approach is asset-based, and [LT]² is a free resource—both support instruction with the potential to disrupt race and class inequities stemming from deficit-thinking and income disparities (McKown and Weinstein 2008).

The free, research-based, [LT]² tool was built for educators, students, and families. [LT]² can assist you in identifying and understanding goals; children's thinking and learning; and teaching children "their way," at the appropriate level, with motivating, playful activities. The asset-based LT approach does not "break down" mathematics into pieces to teach by rote but supports children in building up from their strengths, including the strengths of their families and cultures. LTs and [LT]² can help you to help children find the mathematics in—and develop the mathematics from—their everyday activities, including art, stories, puzzles, and games. Combining close observation of children in the context of these activities with knowledge of goals and developmental progressions allows you to differentiate instruction and continually challenge and support children's mathematical learning.

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