High-Quality Early Math: Learning and Teaching With Trajectories and Technologies High-quality Early Math: Learning and Teaching with Trajectories and Technologies

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Chapter 15
High-Quality Early Math:
Learning and Teaching With Trajectories and Technologies

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ABSTRACT

The importance and complexity of young children's mathematical thinking and learning warrants high-quality, research-based resources that help teachers and caregivers understand and support children's development from birth through the primary grades. The authors discuss young children's potential to think mathematically, the criticality of early math, and the need for a learning trajectories approach to early math. Describing existing risks to young children's experience of high-quality math, the chapter offers solutions to these risks in systematic research and development of technology-based resources for early math using learning and teaching with learning trajectories ([LT]2, at LearningTrajectories.org) as an example. Further, the authors advocate for a lens of equity, inclusion, and accessibility in the development of these technologies. Finally, a vision is described for increasing access to high-quality math through adaptive technologies that use the learning trajectories of early math for in-person and online activities.

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Learning and Teaching with Trajectories and Technologies

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ABSTRACT

The importance and complexity of young children's mathematical thinking and learning warrants high-quality, research-based resources that help teachers and caregivers understand and support children's development from birth through the primary grades. We discuss young children's potential to think mathematically, the criticality of early math, and the need for a learning trajectories approach to early math. We describe the development and early evaluation of the research-based tool Learning and Teaching with Learning Trajectories ([LT]^2, at *LearningTrajectories.org), designed to help early childhood educators, professional development providers, caregivers, and parents to learn about how children from birth to 3rd grade think and learn about math and how to support that learning and development through in-person and online instructional activities with accompanying materials for teaching and learning. Finally, the chapter addresses risks and opportunities of technology-based learning resources for early math and outlines potential directions for future research and development.*

Keywords: Learning Trajectories, Building Blocks, Developmental Progressions, Curriculum Research Framework, Equity, Access, Inclusion, Accessibility

INTRODUCTION

Children need opportunities to think and learn about math deeply and broadly. High-quality early math experiences may be difficult to find amongst widely available platforms such as websites and apps aimed at children. Thus, educators need systematic research and development processes for developing technology-based resources for teaching and learning early math. This chapter briefly describes the research and development of the learning trajectories of early math and details two phases of formative research and development of an online platform for teaching and learning early math. Online platforms aimed at children are examined to further consider opportunities and risks of the development of technology-based resources for early math. Considerations for the quality, accessibility, inclusion, and adaptability of these resources are woven throughout.

BACKGROUND: YOUNG CHILDREN AND MATHEMATICS

With opportunities, young children can learn an informal knowledge of mathematics that is amazingly broad, complex, and sophisticated (e.g., Baroody et al., 2019; Clarke et al., 2006; Clements & Sarama, 2021; Fuson, 2004). For example, toddlers independently enjoy composing three-dimensional shapes (e.g. playing with boxes) or noticing differences in size (e.g. putting baby dolls with a larger mama doll). Preschoolers can learn to invent solutions to solve simple arithmetic problems (Sarama & Clements, 2009a). Also, almost all children engage in substantial amounts of pre-mathematical activity in their free play. They explore patterns, shapes, and spatial relations; compare magnitudes; and count objects. This is true regardless of the children's income level or gender (Seo & Ginsburg, 2004). That is especially important, because all children have the capacity and motivation to mathematize such play and engage in mathematical thinking, but not all are given the opportunity to do so (Engel et al., 2016). Given that early math knowledge is a strong predictor of later achievement (Krajewski, 2005; National Mathematics Advisory Panel, 2008), such learning opportunity gaps are pernicious (Claessens et al., 2007; Horne, 2005; National Mathematics Advisory Panel, 2008).

High-quality education can help children learn to mathematize (Clements et al., 2013; Doig et al., 2003; Thomson et al., 2005). However, if high-quality mathematics education does not start in the earliest years and continue through the early years, children from under-represented groups are often trapped in a trajectory of failure (Clements & Sarama, 2021; Rouse et al., 2005). Typical early childhood classrooms underestimate children's ability to learn mathematics and are ill suited to help them learn. Children may regress on some math skills during pre-K (Farran et al., 2007) and kindergarten (Wright, 1994).

What is high-quality early mathematics education? The recipe has three main ingredients, each grounded in research (Clements, Sarama, et al., 2004; National Research Council, 2009; Sarama & Clements, 2009a).

1. Math content that focuses on big ideas that are mathematically central and coherent (NGA/CCSSO, 2010).
2. A focus on children's ways of thinking and learning as the core of planning and implementing educational experiences (e.g., Carpenter et al., 2014; Gelman, 1979).
3. Teaching practices that honor both the discipline of mathematics and the children's cultures, families, individual characteristics, and patterns of thinking and learning (Carpenter et al., 2014; Frye et al., 2013).

A framework that combines these components is the learning trajectory (LT) approach (Frye et al., 2013; National Research Council, 2009; Sarama & Clements, 2009a). Each learning trajectory includes: (a) a goal; (b) a developmental progression; and (c) linked teaching practices.

Consistent with high-quality mathematics, the LT's goal goes beyond a behavioral objective to include a cluster of concepts, skills, and mathematics practices. Consistent with high-quality early childhood education, the approach puts children's development at the core, and teaching practices are crafted to support each level of this natural development. That is, just as children learn to crawl, then walk, then run, skip, and jump with increasing speed and dexterity – levels of movement – children follow natural developmental progressions in learning math. Teachers who understand the levels of these developmental progressions for each major domain or topic of math, and base their instruction on them, build math learning environments that are particularly developmentally appropriate, effective, and meaningful.

The third component of learning trajectories is the teaching practices that are based on the development progressions and, indeed, are designed specifically to build each level of thinking. Teaching practices are broad, including educational environments, interactions, and activities, matched to each of the levels of thinking in a progression, that help children develop ever-higher levels of thinking. Thus, the learning trajectories approach includes each of three important components of high-quality math.

PROBLEM: Opportunities and Risks in Available Math Apps

Myriad technology games are available. To examine how consistent they are with components of high quality math, 18 apps and websites were chosen through one of two methods: by searching "preschool math" in the App Store to replicate the experience a parent or teacher might use to find an app and through review of published articles on early childhood online math games. When available, games were played and evaluated for content of math within the tasks of the games, information given during the game, feedback following incorrect answers, and scoring, leveling, and reporting features of programs. These included both free apps with in-app purchases or advertisements and apps that require subscriptions. Examples of opportunities and risks from these interactions are provided.

Core Math Content

As stated, technology-based resources for children need to include a developmentally appropriate focus, or goal, of math content as a base for choosing the educational experience we want children to have (NGA/CCSSO, 2010). We used the titles and tasks of game play to determine the intended focus of the app. The most common topic in these titles and descriptions was counting, with 11 of the 18 apps or websites claiming to cover this topic. Many of the games – 10 of the 18 – could also be categorized as covering number comparison. The third most frequent topic was geometry, with 7 programs involving two-dimensional shapes, 1 including three-dimensional shapes, and 3 including some coverage of spatial reasoning. Five apps also involved addition and subtraction. One app and one site also had a game involving three-dimensional shapes. Topics rarely covered included composing numbers, subitizing, measurement, patterns, angles, fractions, volume, and data analysis/classification. Five programs also included non-mathematical games or elements of games. Only one program (Paper Boat Apps, 2021) covered at least one topic within each of the five content areas of mathematics identified by national organizations (NCTM, 2000; NGA/CCSSO, 2010). This reveals the first risk of a crowded and somewhat unregulated market for technology-based learning resources – the breadth of mathematical topics is skewed toward counting and two-dimensional shapes. Many important topics are being missed in early levels of operations and measurement.

Children's Thinking and Learning

In addition to the lack of breadth in the reviewed games, apps, and websites, there is also a lack of depth connected to a rigorous understanding of children's mathematical thinking. When playing these games, many of those claiming to cover these topics do so in a limited sense and others provide experiences that are not mathematical or may confuse children's mathematical understanding.

Although some games were purposefully designed or aligned to the research-validated learning trajectories of early math (Can, 2020; Clements & Sarama, 2021; Ginsburg et al., 2019), most were not. Some sets of games cite learning trajectories as part of the research for developing games, although our interactions did not allow us to determine if and how the developmental progressions would be traversed (Bang et al., 2021; Betts et al., 2020). An opportunity of games is that experiences can be designed to focus on what children need, such as levels of counting, number comparison, or shape learning that may be tedious to adults (Age of Learning, 2021; ToyaTap Ltd, 2016); a risk is that, with a research base, the games may not be calibrated to fine-grained levels of thinking that children need to experience.

Designing games to cover the levels of thinking, technology-based learning resources need to provide experiences that are clear to children and support learning objectives of early math. Unfortunately, some of the available games can be confusing. An example of this was a task asking children to choose all the objects with "one". The "correct" answer included "one" bunch of bananas - the bunch showing three bananas (Tiltan Games, 2013). Other examples included asking children to "tap on three dragons" when the distractor animal (incorrect answer) was a very similar looking dinosaur (Alligator Apps, 2017) . In addition to the wrong answer being easily confused with the "correct" answer, the math contained in the experience is unclear – as any three taps could have met the objective of understanding three as a quantity.

Teaching Practices

Related to a risk of not covering the research-validated levels of children's mathematical thinking (Clements & Sarama, 2021) is the risk of designing games that do not target cognitively distinct levels of math or increase cognitive challenges intentionally, based on diagnoses. For example, many games provided "counting" experiences limited to children tapping the screen and hearing corresponding number words from a game voiceover (Alligator Apps, 2017; Kids Academy, 2020a; Plato Media Ltd, 2021; ToyaTap Ltd, 2016). Although this can be a valuable experience, games or sets of games should provide more challenging tasks so that children can move out of hearing a verbal count and into levels of object correspondence and counting to produce a quantity. Other games included challenges that would usually not be attainable by preschoolers, such as those involving number sentences (4Brains Studio, 2016; Kids Academy, 2020a); using numerals as the primary object for ordinality experiences, without experiences comparing sets or ordering quantities (Age of Learning, 2021; Kids Academy, 2020b; StudyPad, 2019); or providing a science-focused classification task that would necessitate knowledge about animals (Plato Media Ltd, 2021). Because children's math learning and development can have large variation based on their experiences, children's technology-based learning experiences need to be designed to cover the depth of math topics and provide scaffolding to support deep learning and connections.

Furthermore, some games provide experiences of non-mathematical or inaccurate content. Some neutral examples included games that matched pictures of objects or games where the focus is on a sports-related task rather than the skip counting that happens to be occurring on the scoreboard (MIND Research Institute, 2021; Plato Media Ltd, 2021; RosiMosi LLC., 2012; Tiltan Games, 2013). For these kinds of games, the primary risk is that parents or teachers may expect a child to be spending time on mathematical content when they are not. More problematic examples are when the content is not mathematically accurate. A frequent example was the inclusion of non-geometrically defined shapes mixed with geometric shapes (Elamin LTD., 2021; Kids Academy, 2020b; Plato Media Ltd, 2021; RosiMosi LLC., 2012). Although seemingly harmless, a high-quality example of a shape activity included a brief introduction of a square as "A square has four straight sides (sides blinked) and four corners (corners blinked)" before the task of "Find a square." was requested (Khan Academy, 2022). Describing the mathematical properties of a heart or a star in a similar way is not possible, as they are not geometric shapes.

Equally important, providing stories and context can bring joy and meaning to learning. Non-digital instructional activities that provide context, in the form of a storyline, have been shown to be more effective at improving children's math learning than those without context, particularly for children with lower executive function (Veraksa et al., 2020). Games that include a context for the mathematical tasks are likely to be more engaging and effective as well. However, many games provided little context, such as a shape game in which a truck appeared and children could match pictures of shapes on one side of the screen to the shapes on the truck – a goal that needed to be accomplished through guessing, as no storyline or directions were provided through a voiceover (Kids Academy, 2020b). Other apps provided experiences similar to flashcards or worksheets, with no context in which children could situate the materials (4Brains Studio, 2016; StudyPad, 2019).

Finally, it is important to note that many of the same risks and opportunities involved in the education and care of young children exist across digital and in-person contexts. For example, the use of violence should be avoided in content aimed at young children, such as animating a bunny getting hit with a club as a context for one-to-one correspondence (Tiltan Games, 2013). Similarly, most modern educators understand that a practice to avoid is feedback that sends a message about children's innate abilities, such as "You are brilliant!" when children get correct answers (Paper Boat Apps, 2021). Instead, we have an opportunity to use technology to promote positive dispositions toward persistence in math by using feedback focused on effort or the accomplishment of challenging tasks (Dweck, 2017).

Reviewing these easily available technology-based resources that are intended to support early math revealed that many risks exists to providing children with high quality early math experiences. Although interventions are more effective if they involve families, especially by providing activities parents can do with their children (Halpern, 2004; Ramey & Ramey, 1998), many families and teachers have a limited view of what math is appropriate for young children (Sarama, 2002). This difficulty of how to teach math may be encumbered by difficulties with math in general, as the National Council for Education Statistics report that 30% of adults in the U.S. struggle with numeracy (Mamedova & Pawlowski, 2020). Thus, ameliorating the many risks to high quality math experiences in technology necessitates rigorous and zealous new approaches to early math technology-based resources.

SOLUTIONS AND RECOMMENDATIONS

Opportunities for learning math are maximized when technology-based resources include the three components described at the beginning of this chapter – core math content, a focus on children's thinking and learning, and teaching practices that follow developmental progressions and focus on math as a discipline and children as individuals with unique cultures, families, characteristics, and abilities. We advocate for ensuring the inclusion of these essential aspects within technology-based resources through systematic research and development processes, consideration of diversity and inclusion, and by ensuring resources are equitably accessible to a broad audience.

The research and design of the Learning and Teaching with Learning Trajectories, $[LT]^2$**,** tool, launched in 2020 following 25 years of development, was designed to meet this need. $[LT]^2$ provides high quality math with content and resources across 20 math topics. Resources provide content information, text and videos sequences, and describe and embody children's development of progressive levels of mathematical thinking. Activities provide concrete examples of content aligned to developmental progressions. They are written with key questions and suggestions for individualizing the activity to children's needs. Text, pictures, and videos provide concrete examples of children's development, as well as teaching and learning contexts.

SOLUTION 1: DEVELOPMENT OF TECHNOLOGY FOR EARLY MATH LEARNING

As a first step to providing innovative solutions to the problem of widely available technology resources that do not meet basic expectations of high quality early math, we advocate for systematic research and development within a framework focused on comprehensive content, an understanding of children's thinking and learning, and efficacious teaching practices. Specifically, we describe the development and research of both the learning trajectories approach and its current technology-based resources – the $[LT]^2$ tool.

Development of Early Math LTs

How were the learning trajectories developed? To ensure that they would be research-*based* and research-*validated,* the authors created and applied a Curriculum Research Framework (CRF) that includes ten phases embedded within three categories (Clements, 2002, 2007). The first category, a priori foundations, included reviewing and synthesizing all the available research, thousands of studies and documents from around the U.S. and around the world (Clements, Sarama, et al., 2004; Sarama & Clements, 2009a, 2019). This synthesis created the first draft of the learning trajectories.

Goals

Based on both the expertise of mathematicians and research on students' thinking about and learning of mathematics (Clements, Sarama, et al., 2004; Fuson, 2004; National Mathematics Advisory Panel, 2008; Sarama & Clements, 2009a), LT goals are organized into the "big" ideas of mathematics: overarching clusters, concepts, and skills that are mathematically central and coherent, consistent with students' (often intuitive) thinking and learning, and generative of future learning (Clements, Sarama, et al., 2004; NCTM, 2006) such as "numbers can be used to tell us how many, describe order, and measure" and "geometric shapes can be described, analyzed, transformed, composed, and decomposed into other shapes" (Clements, Sarama, et al., 2004).

Developmental Progressions

Children follow a developmental progression of levels of thinking toward understanding and competence in a particular mathematics topic (Clements, Sarama, et al., 2004; Fuson, 2004; Gravemeijer, 1999; Sarama & Clements, 2009a). Such levels are well developed and validated for counting (e.g., Clements & Sarama, 2021; Fuson, 1988; Steffe & Cobb, 1988), arithmetic (e.g., Carpenter et al., 2014; Sarama & Clements, 2009a), and even geometric measurement (Barrett et al., 2017; Stephan et al., 2001).

An example in a different domain, composing and decomposing shapes, was generated from research from Australia and the U.S. that was combined to provide a framework for children's levels of thinking (see Clements, Wilson, et al., 2004). Children initially treated shapes such as pattern blocks separately, then they combined them, but each shape was a separate part of a picture. Later, they began laying some shapes side-by-side with trial and error, then they also started recognizing and planning to combine shapes to make other shapes. Studies then supported each level of this developmental progression (Clements, Wilson, et al., 2004). The current conceptualization of the learning trajectories for early mathematics includes 222 levels of thinking across 20 topics. Research continues to support the refinement of these topics and levels.

Instructional Practices

The third part of a learning trajectory consists of sets of teaching practices–environments, interactions, and activities–fine-tuned for each level of children's understanding to help them become proficient in that level before moving on to the next level (Clements & Sarama, 2021). The teaching practices are designed to help children learn the ideas and practice the skills needed to master that level. For some topics, teaching experiments (Steffe et al., 2000) and other types of studies provided guidance and even specific activities to include (e.g., Carpenter et al., 2014; Gravemeijer, 1999; Sarama et al., 2021; Stephan et al., 2001). For others, such as shape composition, the existing research only gave hints as to effective practices, although a sequence of puzzles that required each successive level of thinking form a reliable core of activities. The later, especially, needed to be refined and validated.

Refinement and Validation of the Learning Trajectories

The second category of the CRF – the learning model – tested and refined the learning trajectories, and even developed them when existing research was not available. In this intensely iterative process, learning trajectories were applied and improved dynamically, using grounded theory methods, clinical interviews, teaching experiments, and design experiments (Clements, 2002, 2007; Ginsburg, 1997; Sarama & Clements, 2004; Steffe et al., 2000). As an example from shape composition, early versions of puzzles on computers found that children learned less from direct manipulation of shapes, such as dragging to slide or turn, than they did from selecting slide and turn tools (Sarama & Clements, 2002). Using the tools, they became explicitly aware and learned to differentiate the three geometric motions and even quantified them ("I gotta turn it 3 times!"). In all cases, qualitative research showed the efficacy of each of the learning trajectories.

In the third category of the CRF – formative evaluation – researchers collect empirical evidence to evaluate appeal, usability, and effectiveness, revising the learning trajectories multiple times (e.g., Sarama, 2004; Sarama & Clements, 2004). Finally, summative research showed strong effect sizes between 1 and 2 standard deviations (Cohen's d). The approach increased the quantity and quality of the math environment and teaching, and substantially increased achievement (Clements & Sarama, 2008). Our example topic, shape composition, showed one of the largest relative gains. These strong, reliable benefits of a learning-trajectories curriculum have been substantiated by other researchers (Anthony et al., 2011; Weiland & Yoshikawa, 2012), including in other countries such as Ecuador (Bojorquea et al., 2018). Also, other curricula and projects also grounded in learning trajectories have had success (e.g., Griffin, 2009; Perry et al., 2008; Wright et al., 2006; Young-Loveridge, 1989). This strongly validates the learning trajectories in $[LT]^2$.

Research and Redesign of an Online Resource

Teachers in many of the aforementioned studies developed an understanding of learning trajectories using an early website, Building Blocks Learning Trajectories (BBLT) that included descriptions and videos. The new Learning and Teaching with Learning Trajectories, $[LT]^2$, tool is an updated and a greatly expanded version of BBLT. $[LT]^2$ provides interactive ways of defining goals, identifying important aspects of children's mathematical thinking, and teaching children at their level.

This section explains the processes and evidence used to ensure the quality of $[LT]^2$ in relation to its use in curriculum, assessment, and professional development.

The research and design efforts of $[LT]^2$ itself have spanned seven years and improvements are continuing. Alhough the first two phases of the CRF were completed through previous work, the creation of a new platform required new phases of formative and summative assessment. We detail two phases of formative research and design efforts aimed at broadening access to high quality early math resources.

Formative Research – Phase 1 Given the success of BBLT in research projects (Clements & Sarama, 2008; Clements, Sarama, Spitler, Lange, et al., 2011; Clements et al., 2015; Sarama et al., 2016), the first goal was to learn which aspects of BBLT were considered most useful and which needed improvement. Respondents also indicated that the tool was used to learn about math topics and to learn about children's mathematical thinking. BBLT was ranked most useful for instruction ($\bar{x} = 5.28$), while the feature "test yourself" allowing users to check their understanding of the learning trajectories was rated least useful ($\bar{x} = 4.83$). Thus, although variation existed, the lowest and highest rankings were still represented a majority of users rating the features positively. Qualitative data revealed that the most frequently liked aspect of BBLT was the videos. Thus, updating the videos became a prime target for the new $[LT]^2$ tool.

The design of $[LT]^2$ began with the feedback from BBLT users, and a new site was available for testing in 2015. This new site was evaluated through surveys of early childhood professionals and intensive interviews. The early childhood professionals worked in center-based childcare, Head Start, and a private preschool. Activities linked to the developmental progressions were most frequently rated as being useful for the classroom. Interviewers sat next to each person as they used the site, exploring it freely or trying to find information or perform tasks suggested by the interviewer (e.g., "What is it you wanted to learn? What level of thinking are your students probably working at?"). Feedback from these users provided information for refinement of the site, particularly the need for videos explaining each trajectory as a whole. As details for the site were being constructed and refined, new online software for children was also being developed for subitizing, counting, and shape composition.

A primary aim for the new site was to add exemplary instructional and developmental videos. Introductory videos were also created for each math topic. Many new videos were generated in 2015 and 2016, and this process continues with contributions from collaborators. Games were tested and updated to clarify the logic and to improve motivational aspects through better graphics and sound. This improved the quality and usability of content available through the site.

In 2017, project staff conducted and applied reviews of research on infant and toddler math from a learning trajectories framework. Thus, the original LTs covering ages three to five, now cover birth to age 8, in both the description of typical development and suggestions for activities to support learning and development across these age ranges. This increased the range of children and caregivers who could benefit from the platform.

Another important new feature implemented in 2017 was the alignment of the learning trajectories to national and state standards, as well as to the commonly used early childhood assessment system – Teaching Strategies GOLD. This work was incorporated into what is now the Focus Tool, allowing users to focus on the age/grade, mathematical strand, standard, or assessment that will help them identify a level that is their goal for planning math experiences for young children. Although new state standards and updates to the learning trajectories will impose ongoing updates, the Focus Tool is now available for the majority of standards and math topics. Thus, the first phase of formative research ensured that the technology included math content was broad and comprehensive and supported educators' planning aligned to children's learning and thinking.

Formative Research – Phase 2

Given the significant changes in the first phase of formative research and new elements, a second phase was conducted. We conducted both formal and informal testing of the new site. Informal testing took place through feedback from partners through professional meetings such as the National

Association for Young Children (NAEYC) and the National Council of Teachers of Mathematics (NCTM). Through this feedback, several resources were developed: a privacy policy, a user guide, and a dedicated email address to communicate with users. These collaborations also resulted in new video being received from collaborators.

Participants in the usability study indicated that the filters and video were helpful for their planning, as well as key questions embedded in the instructional activities. For implementation, participants indicated that they used the activities as a context for formative assessment and that $[LT]^2$ improved their mathematical content knowledge. Finally, for reflection, participants indicated that the tools for tracking individual progress were helpful and that revisiting the reports from this tool assisted with confirming other evidence of children's level and/or in changing their beliefs about children's level. General themes included that the site was less overwhelming than the book and that the videos made the content and concepts more concrete.

Users asked for instructional activities to have clearer definitions, more materials, and additional support for questioning and differentiation. They also asked for more videos and for video descriptions to highlight child thinking, verbalization, and gesture. Finally, they asked for more resources to understand contextual skills, the role of language, and supports to engage families or allow families to independently engage with the content. As a result, the second phase of formative research informed the generation of resources for teaching practices that honor the discipline of mathematics and children's cultures and families.

Online software for children (games) were tested for logic and motivation and later tested for their contribution to children's learning. To assess game-based learning, a teaching experiment was conducted in 2018 (Steffe et al., 2000). The design of this study included six sessions with six students. Each student was given pre- and post-assessments in the topics of math related to game play. Instructors also facilitated activities before and after game play each session to ascertain transfer of concepts from game play to non-digital contexts. Of the 6 students in the experiment, 4 increased levels from pre- to post-assessment in 2D Shape Composition, 3 increased in 2D Shapes, 5 increased in Counting, and 4 increased in Subitizing. Thus, the Number Path game "Help the Turtle Get Home" was the most successful in supporting learning. Shape Puzzles was the least successful in supporting learning specific to the 2D Shapes trajectory (not the primary goal), but it was more successful in supporting learning for 2D Shape Composition. Testing of the online games for children validated the usefulness of the games to support learning through pre- and post-assessments. However, one game was more successful in supporting learning for shape composition than an understanding of shape matching, identification, and classification. Thus, the development of a new game that more specifically targets shape matching and identification is currently in progress.

Launching and Maintaining

The newest version of the site was launched in February of 2020. It now has over 30,000 registered users across 38 countries. The user base has grown rapidly since the beginning of the COVID-19 pandemic. Project staff have responded by producing a quick start guide and adding activities with tips for social distancing or online learning. The next phase – summative evaluation – is underway now, funded by the National Science Foundation.

New research on young children's math learning and development informs updates to the topics, levels, and activities on the site. A synthesis of recent research was used to update all topics and levels during the summer of 2020 (Clements & Sarama, 2021). As new math lessons, activities, and experiences are tested for their effectiveness, project staff are evaluating this work and considering whether and how they fit within the LT framework. This free resource will continue to provide content for professional development and implementation of developmentally appropriate early math experiences. The 7-year process of building the site has culminated in a useful tool for anyone interested in supporting the mathematical learning and development of young children by defining goals, identifying important aspects of children's mathematical thinking, and providing high quality instruction for teaching and learning math.

Solution 2: Equity, Inclusion, and Accessibility

As previously discussed, an important component of high-quality math is teaching practices that honor children's cultures, families, individual characteristics, as well as their patterns of thinking and learning (Carpenter et al., 2014; Frye et al., 2013). This can be particularly challenging in the context of technology, as these personal characteristics are most sensitively responded to in the context of personal relationships. In this section, we describe promising research and development efforts to advocate for pursuing these challenges to provide math experiences for today's global citizens, situated in social contexts in which education is at the forefront of valuing and bolstering a more pluralistic society. We connect this to new features on [LT]².

The United States has been experiencing a double pandemic – a virus attacking respiratory systems and a growing awareness of long-standing inequities in educational resources, exacerbated by COVID-19, for Black, Indigenous, and people of color (BIPOC) and children with disabilities. Most young children and their teachers do not have access to empirically validated early childhood mathematics materials (Clements & Sarama, 2021) and both the income gap and the associated opportunity-to-learn gap have been increasing for decades (Bachman et al., 2015). During a time when access to such materials may be even more limited due to health concerns, inequities are further magnified (Kuhfeld et al., 2020). Thus, technology-based learning resources could be part of this ongoing risk to vulnerable children and families if they are not financially accessible, exclusive of BIPOC, women, and people with disabilities, or maintain inequitable power dynamics in school-family partnerships. In contrast, technology-based learning resources have the potential to disrupt these inequities if they are adaptive, inclusive, and accessible to diverse children and families.

Equity

Evidence suggests that computer experiences can reinforce math concepts taught in classroom settings (Clements, Sarama, Spitler, Lange, et al., 2011; Clements, Sarama, Spitler, & Wolfe, 2011; Jenkins et al., 2018). In one study of 36 classrooms, the number of computers running math software was one of the variables that most differentiated the treatment and control groups - correlating with child gains (Clements & Sarama, 2008). In a study of 106 teachers, the classroom observation indicated that the number of computers running *Building Blocks Software* had one of the three highest correlations with child outcomes - with gains in both math and expressive language (Sarama & Clements, 2009b). Studies also show that preschoolers can make significant learning gains with software targeted to specific mathematics topics (Brinkley & Watson, 1987-88; Clements & Sarama, 2003; Hungate, 1982), especially when they include computer manipulatives (Clements & Sarama, 1998; Ishigaki et al., 1996; Sarama et al., 1996). Significant positive effects on mathematics achievement were also found for kindergartners, compared to language and literacy software (Anthony et al., 2011).

Further, Black children gained significantly more than others in some math interventions, closing an initial gap (Clements, Sarama, Spitler, Lange, et al., 2011) with impact persisting for years (Clements et al., 2021; Clements et al., 2013). An emphasis on procedural rather than conceptual learning and lower expectations of teachers for children of color has been suggested as a reason preschool benefits dissipate for Black children (Currie & Thomas, 1995; Zhai et al., 2012). In contrast, interventions incorporating learning trajectories, including computer applications, have been particularly effective with children of color and dual language learners (Clements, Sarama, Spitler, Lange, et al., 2011; Foster et al., 2018) because they adapt to each child's strengths and needs. These promising results are evidence that technology can increase equitable learning opportunities for young children's math learning.

Inclusion

Intentional efforts to make participation inviting to diverse populations can increase the impact of technology-based learning resources, as it may increase the reach of such resources. Inclusion can begin with ensuring that images are representative. Similarly, references to holidays, foods, and other cultural artifacts are not centered on one group. This can be improved by checking bias for cultural centricity of one's own group, as well as intentionally including team members with diverse perspectives. An area for further research and innovation is to study the extent to which inclusion efforts encourage and sustain engagement of diverse groups in technology-based resources for early math.

A new game in [LT]^2 was designed to follow the shape LT, and also to explicitly include people of color, people with disabilities, and women. The game begins with a shape puzzle focused on matching or recognizing shapes. When the puzzle is complete, the object animates to reveal a carefully chosen picture – such as a space shuttle. This space shuttle is then shown next a person who contributed to space technology, such as Katherine Johnson, and a brief bio is read. Each person in this game was chosen to provide a positive and empowering example of scientists, engineers, inventors, and mathematicians who are also women, people of color, or people with disabilities. These STEM heroes are highlighted to expose children to a diverse view of who has contributed to STEM – and our hope is that many children see themselves in these games and know that they can also choose to become future scientists, engineers, inventors, and mathematicians. Including games like this one within a system based on learning trajectories is a piece of the vision for a future of high-quality math for all children.

Accessibility

Beyond inclusion, accessibility for people with disabilities is another important factor in ensuring a future of high-quality math for all children. Within technology-based learning resources, accessibility ensure that people with different abilities can participate in learning activities. This includes ensuring that children and the adults in their lives can perceive, understand, navigate, and interact with the technology.

For example, accessibility features on the [LT]^2 website include closed captions for videos to increase the accessibility of video content for people who are deaf or hearing-impaired. Another feature is a second set of videos that have an audio overlay describing the important mathematical actions that occur in videos. This feature was designed to assist people who are blind or vision-impaired to access the content of the site. An accessibility widget is accessible through keyboard shortcuts for those who use screen reader technology and can turn on voiceover videos across all pages, change the color contrast of all pages, or increase the font size on all pages. These features increase the accessibility of content for adult learning.

Because the [LT]^2 technology also includes content for children – both digital games and in-person activities implemented by adults – [LT]^2 project staff have partnered with colleagues at the center for STEM Innovation for Inclusion in Early Education (STEMI^{2}E^2) to develop early math instructional experiences with accommodations for diverse learners and to test these with children learning in inclusive settings. Further, staff have created general accommodations for teaching math with a focus on accessible environments, materials, and instruction. First, ensuring an accessible environment includes creating space in which children can move freely to their capacity. Second, ensuring accessible materials involves considering differences in how children would interact with math manipulatives and other learning materials. Third, ensuring accessible instructional processes means considering and accommodating ways in which children may respond to and receive information from learning experiences differently. Staff have also created recommended accommodations for each level of each math topic across developmental domains of cognition, motor, sensory, communication, and social emotional needs. These resources are primarily focused on in-person learning. With programming partners, innovative efforts to ensure universal access include creating digital experiences for children with contrasting colors and touch-screen options. Digital games for children can also be programmed with enhanced accessibility through unique display options such as feedback using symbols or emojis and audio settings that change based on children's sensitivity to noise. In conclusion, all aspects of a technology-based system for early math should be accessible – the resources for adult learning, the in-person instruction for children, and the digital math experiences for children.

Solution 3: More Accessible and Adaptable—Present Directions

Throughout the launch of versions of the **[LT]²** tool, highly engaged users celebrated the plethora of resources and activities. Other users lamented that the resources seemed difficult to access due to the need to understand the developmental progressions, assess the level of thinking at which their children might currently be working, and then choose activities that supported and scaffolded that level of thinking. Many adults wanted to jump to activities – which misses the opportunities of using the learning trajectories approach that combines goals, developmental progressions, and instructional activities. Thus, a next step in efforts to increase high quality math experiences for all children is to build technology that is more accessible and adaptable to anyone interested in supporting young children's math learning.

The system that is envisioned would provide a starting place for adults based on children's age, recommend in-person activities as well as digital games for children, assess children's level of thinking based on games, and recommend next steps for in-person activities based on children's level of thinking. Providing a path that outlines where to start and where to go based on interactive with games that assess and level will increase the ease of access to high quality math regardless of race, ethnicity, gender, ability, socioeconomic status, or other personal characteristics. Having translated versions of resources within the same framework would also remove the barrier of language.

As an example, the system would begin with information about the child's age and a topic. If a child were preschool age and the adult were interested in geometry, the system would start with investigations of two-dimensional shapes at the level: *Shape Recognizer – Circles, Squares, and Triangles*. If the adult user identified themselves as a family member, a playful game such as "Shape Hunt (Triangles)" would be recommended. In this game, family members show the child an accurate triangle, hide triangles around a room, and encourage the child to find as many as they can. If the adult user is a teacher, a recommended activity would be "Is It or Not? [Shape Recognizer (Circles)]." In this instructional activity, often completed in whole group, a teacher draws shapes while children watch and asks them to decide whether it is or is not a circle – and explain why! Also at this level, both teachers and families can encourage their children to play the digital game – "Hidden Pictures – Shape Matcher 2" which challenges children to complete a puzzle by matching individual shapes. Children's level of success on the "Hidden Pictures" game would estimate the extent to which they are successfully recognizing circles, squares, and triangles. If the game estimates that they are ready to move to a more challenging, level of math, the system would then recommend that teachers or families try a game such as "Build Shapes" at the level: *Constructor of Shapes from Parts*. This iteration of games-based learning and assessment with challenging in-person activities would continue along the developmental progressions, with instructional activities designed to continuously advance children's learning.

Thus, an adaptable and intelligent path will provide adults with a hint about how to begin challenging their children's early math learning. Then, an intelligent technology-based system would be able to incorporate information from children's game play to estimate children's level of thinking. Finally, this innovative system would create iterative feedback loops to inform adults of additional instructional activities to implement in the context of their personal relationships with children. To summarize, an innovative approach to ensuring children have high quality math experiences is to intentionally combine personal and digital experiences in response to children's levels of thinking and learning.

Adaptability

This combination of digital and personal experiences with math, adapted to children's needs, can be supported through adaptive games. Games development, in addition to being focused on learning and engagement, can also be developed from a measurement perspective, particularly those with embedded assessment or adaptive leveling. One method for this is developing a table of specification of the content or cognitive processes of interest, types of items that would access that process or knowledge, and the number of items needed to generate information about that knowledge (American Educational Research Association, 2014). Other methods include testing games with children as described previously in this chapter – comparing children's progress on games with their progress on non-digital assessment tasks. Observations with think-alouds are frequently used in usability testing of websites – a similar method are cognitive interviews in the development of surveys (Nielsen, 2012; Ryan et al., 2012). Both methods serve to understand how research participants perceive the experience by asking for them to talk about their interpretations and interactions throughout the experience. Other methods to ensure games are clearly following an intentional leveling include developing a matrix defining the ways in which tasks should vary by cognitive challenge and representations (Henson et al., 2009). Like assessments, games can also be a series of challenging tasks that children achieve or not, with the gaming element emphasizing a satisfying end to the quest or series of tasks. If game data are used to evaluate and progress children's learning, the elements of games that lend themselves to assessment should also be developed and evaluated with methods appropriate to the development assessments embedded within adaptive sets of early math games.

Technology-based approaches to early math provides excellent opportunities to provide content within children's zone of proximal development (Vygotsky, 1935/1978) by differentiating games-based instructional tasks through embedded assessment and dynamic leveling. Future research can more specifically develop and test the use of adaptive technologies to understand and progress children through developmental progressions. Although many of the games include a path or sequence in which the games are given, some had more clarity in how the path would progress, such as adapting to learners' needs by analyzing information from interaction with games (Betts et al., 2020). The rationale for other sets of games was unclear both in the choice of games and the path, such as one that included angle learning in a game aimed at toddlers (Kids Academy, 2020b) Others had progressions that were more aligned to research, such as providing games aligned to age-groupings, with games providing experiences up to the Patterner AB level for children in PreK, while activities including AAB patterns and Growing Patterns are in a section of games for Kindergarten (Paper Boat Apps, 2021). Because experience is a large part of what drives children's mathematical learning, allowing children to access content that is above or below what is typically considered grade-level material creates important opportunities. This phenomenon was seen in a study of an adaptive set of math games that were less beneficial for children with low knowledge at the beginning of the study (Betts et al., 2020). Another study of an intervention with a set of digital games showed a negligible effect on math scores, with authors hypothesizing that the lack of access to below grade content was one explanation for the poor results (Rutherford et al., 2014). Thus, generating games that are adaptive *and* provide access to experiences at all levels of children's mathematical thinking can create opportunities for all children to learn math. Finally, sequences can be of many types, such as simple sequences of paired associates, Bloom's taxonomy, rational analyses, and others. We have argued for the superiority of developmental progressions of learning trajectories (Clements & Sarama, 2014).

CONCLUSION

High quality math needs to be readily available in a research framework.

The research and development of high-quality early math learning resources is supported through systematic research, such as the phases of the CRF. These processes ensure that products are usable and useful. Similarly, intentionally developing technology-based resources that disrupt inequitable opportunities, welcome and include diverse groups, and accommodate the needs of people with different abilities will broaden the audience and increase the impact of high-quality math technology. Further, innovations for the next set of high-quality math technologies will build adaptable systems that essentially support accessibility to wide audiences, removing barriers through digital assessment and prioritizing math instruction through personal relationships and digital games. Across these efforts, we recommend a learning trajectories approach that challenges children and provides meaningful experiences in both personal and digital contexts.

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KEY TERMS AND DEFINITIONS

Learning Trajectories: An approach to teaching and learning involving the integration of a learning goal, a developmental progression, and aligned instruction.

Developmental Progression: Paths of learning and ways of thinking through which children typically develop.

Building Blocks: An evidence-based curriculum that utilized the learning trajectories approach.

Curriculum Research Framework: A method for developing curriculum based on ten phases across categories of a priori foundations, a learning model, formative evaluation, and summative evaluation.

Equity: The condition of fairness and justice based on everyone getting what they need, often contrasted with equality, a condition in which everyone gets the same thing regardless of need.

Inclusion: An outcome of intentional efforts to make participation inviting to diverse populations in which such populations feel welcome and comfortable.

Access: Applied universally, access is an outcome in which race, ethnicity, language, gender, ability, socioeconomic status, or other personal characteristics do not limit the opportunity to resources such as learning resources.

Accessibility: Applied to people with disabilities, accessibility allows people to participate with activities and content, including being able to perceive, understand, navigate, and interact with technology.